Complex Analysis: Resit Exam

Aletta Jacobshal 02, Thursday 13 April 2017, 18:30–21:30
Exam duration: 3 hours

Instructions — read carefully before starting

- Write very clearly your **full name** and **student number** at the top of the first page of your exam sheet and on the envelope. **Do NOT seal the envelope!**
- Solutions should be complete and clearly present your reasoning. If you use known results (lemmas, theorems, formulas, etc.) you must explicitly state and verify the corresponding conditions.
- 10 points are "free". There are 6 questions and the maximum number of points is 100. The exam grade is the total number of points divided by 10.
- You are allowed to have a 2-sided A4-sized paper with handwritten notes.

Question 1 (30 points)

Consider the function

$$f(z) = ze^{iz}.$$

(a) (8 points) Prove that

$$f(z) = e^{-y}(x\cos x - y\sin x) + ie^{-y}(y\cos x + x\sin x),$$

where $z = x + iy$.

(b) (8 points) Prove, using the Cauchy-Riemann equations, that $f(z)$ is entire.

(c) (6 points) Compute the derivative of $f(z)$.

(d) (8 points) Prove that the function

$$u(x,y) = e^{-y}(x\cos x - y\sin x),$$

is harmonic in $\mathbb{R}^2$ and find a harmonic conjugate of $u(x,y)$.

Question 2 (15 points)

Evaluate

$$\text{pv}\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}dx$$

using the calculus of residues.

Question 3 (10 points)

Use Rouché's theorem to show that, if $0 < \varepsilon < 7/4$, then the polynomial $P(z) = z^3 + \varepsilon z^2 - 1$ has exactly 3 roots in the disk $|z| < 2$.

Question 4 (15 points)

Represent the function

$$f(z) = \frac{z}{z^2-1},$$

(a) (8 points) as a Taylor series around 0 and find its radius of convergence;

(b) (7 points) as a Laurent series in the domain $|z| > 1$.

Question 5 (10 points)

Consider the functions

$$f(z) = \frac{\sin z}{z} \quad \text{and} \quad g(z) = e^{1/z}.$$

Determine the singularities of $f(z)$ and $g(z)$, and their types (removable, pole, essential; if pole, specify the order). Make sure to justify your answer.

Question 6 (10 points)

Consider a function $f(z)$ such that $\text{Re}(f(z)) \geq M$ for all $z \in \mathbb{C}$, where M is a real constant. Prove that if $f(z)$ is entire then it must be constant. *Hint: consider the function* $e^{-f(z)}$.